HEATING OF A CYLINDRICAL BODY WITH INTERNAL HEAT SOURCES AND A VARIABLE HEAT TRANSFER COEFFICIENT

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An approximate method is described for solution of the problem of the temperature distribution in an infinite cylinder when the coefficient of external heat transfer is a function of time. A constant-power heat source acts in the body during the entire period of heating.

The process of heating an infinite cylinder in a medium with a variable coefficient of convective heat transfer and a constant-intensity heat source inside the body is described by the following system of equations:

$$\frac{\partial \theta(\psi, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 \theta(\psi, \mathrm{Fo})}{\partial \psi^2} + \frac{1}{\psi}\frac{\partial \theta(\psi, \mathrm{Fo})}{\partial \psi} + \mathrm{Po}, \quad (1)$$

$$\frac{\partial \theta(1, \mathrm{Fo})}{\partial \psi} = \mathrm{Bi}(\mathrm{Fo})\,[1-\theta(1, \mathrm{Fo})], \quad (2)$$

$$\frac{\partial \theta(0, \mathrm{Fo})}{\partial \psi} = 0, \quad (3)$$

$$\theta(\psi, 0) = \theta_0. \quad (4)$$

We will seek a solution to problem (1)–(4) in the form

$$\theta(\psi, \mathrm{Fo}) = 1 + \frac{\mathrm{Po}}{4}\left[1-\psi^2+\frac{2}{\mathrm{Bi}(\mathrm{Fo})}\right]\varphi(\mathrm{Fo})\,\frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})} \times \exp - \int_0^{\mathrm{Fo}} \frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})}\,d\mathrm{Fo} - \sum_{n=1}^{\infty} A_n\left(1-\theta_0+\frac{\mathrm{Po}}{\mu_n^2(\mathrm{Fo})}\right) Z_n(\mathrm{Fo}) \times J_0[\mu_n(\mathrm{Fo})\psi]\exp - \int_0^{\mathrm{Fo}} \mu_n^2(\mathrm{Fo})\,d\mathrm{Fo}. \quad (5)$$

If the time-dependent roots $\mu_n(\mathrm{Fo})$ are defined by the transcendental equation

$$\mu(\mathrm{Fo})\; J_1[\mu(\mathrm{Fo})]/J_0[\mu(\mathrm{Fo})] = \mathrm{Bi}(\mathrm{Fo}), \quad (6)$$

or

$$J_0[\mu(\mathrm{Fo})]/J_1[\mu(\mathrm{Fo})] = \mu(\mathrm{Fo})/\mathrm{Bi}(\mathrm{Fo}), \quad (6')$$

and the constant coefficients A_n are found from

$$A_n = 2J_1[\mu_n(0)]/\mu_n(0)\{J_0^2[\mu_n(0)] + J_1^2[\mu_n(0)]\},$$

then relation (5) will satisfy the condition on the surface (2), and the condition of symmetry of the temperature field (3).

Functions $\varphi(\mathrm{Fo})$ and $Z_n(\mathrm{Fo})$ may be determined by requiring the best approximation of solution (5) to the differential equation (1). In order for (5) to satisfy the initial temperature distribution (4), the following initial conditions should be satisfied:

$$\varphi(0) = [4+\mathrm{Bi}(0)]/8\mathrm{Bi}(0), \quad (7)$$

$$Z_n(0) = 1. \quad (8)$$

If we substitute (5) into (1) and cancel like terms, we obtain

$$U(\psi, \mathrm{Fo}) + W(\psi, \mathrm{Fo}) = P(\psi, \mathrm{Fo}), \quad (9)$$

where

$$U(\psi, \mathrm{Fo}) = \frac{\mathrm{Po}}{2}\left\{\frac{8\mathrm{Bi}^1}{\mathrm{Bi}(4+\mathrm{Bi})} - \frac{1}{2}\left(1-\psi^2+\frac{2}{\mathrm{Bi}}\right)\left[\frac{32\mathrm{Bi}^1}{(4+\mathrm{Bi})^2} - \frac{64\mathrm{Bi}^2}{(4+\mathrm{Bi})^2} + \frac{8\mathrm{Bi}}{4+\mathrm{Bi}}\,\frac{\varphi^1}{\varphi}\right] - \frac{16\mathrm{Bi}}{4+\mathrm{Bi}} + \frac{2}{\varphi}\exp\int_0^{\mathrm{Fo}} \frac{8\mathrm{Bi}}{4+\mathrm{Bi}}\,d\mathrm{Fo}\right\} \times \varphi\exp - \int_0^{\mathrm{Fo}} \frac{8\mathrm{Bi}}{4+\mathrm{Bi}}\,d\mathrm{Fo}, \quad (10)$$

$$W(\psi, \mathrm{Fo}) = \sum_{n=1}^{\infty} A_n\left\{\left(1-\theta_0+\frac{\mathrm{Po}}{\mu_n^2}\right)\left[Z_n^1 J_0(\mu_n\psi) - Z_n\mu_n^1\psi J_1(\mu_n\psi)\right] - Z_n\frac{\mathrm{Po}\,\mu_n^1}{\mu_n^3}J_0(\mu_n\psi)\right\}\exp - \int_0^{\mathrm{Fo}} \mu_n^2\,d\mathrm{Fo}. \quad (11)$$

We find the mean values of functions $\overline{U}(\mathrm{Fo})$ and $\overline{W}(\mathrm{Fo})$ over the cross section of the cylinder from the equations

$$\overline{U}(\mathrm{Fo}) = 2\int_0^1 \psi U(\psi, \mathrm{Fo})\,d\psi, \quad (12)$$

$$\overline{W}(\mathrm{Fo}) = 2\int_0^1 \psi W(\psi, \mathrm{Fo})\,d\psi. \quad (13)$$

Following substitution of (10) and (12) into (11) and (13), we have

$$\overline{U}(\mathrm{Fo}) = 2 - 2\varphi^1\exp - \int_0^{\mathrm{Fo}} \frac{8\mathrm{Bi}}{4+\mathrm{Bi}}\,d\mathrm{Fo}, \quad (14)$$

$$\overline{W}(\mathrm{Fo}) = \sum_{n=1}^{\infty} A_n\left\{\left(1-\theta_0+\frac{\mathrm{Po}}{\mu_n^2}\right)\left[Z_n^1\frac{2J_1(\mu_n)}{\mu_n} - Z_n\mu_n^1\left(\frac{4}{\mu_n^2}J_1(\mu_n) - \frac{2}{\mu_n}J_0(\mu_n)\right)\right] - \frac{2\mathrm{Po}\,\mu_n^1}{\mu_n^4}Z_nJ_1(\mu_n)\right\}\exp - \int_0^{\mathrm{Fo}} \mu_n^2\,d\mathrm{Fo}. \quad (15)$$

Applying the condition

$$\bar{U}(\mathrm{Fo}) = 0, \tag{16}$$

and taking account of (7), we obtain the following form for the function $\varphi(\mathrm{Fo})$:

$$\varphi(\mathrm{Fo}) = \frac{4+\mathrm{Bi}(0)}{8\mathrm{Bi}(0)} + \int_0^{\mathrm{Fo}} \left[\exp \int_0^{\mathrm{Fo}} \frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})} d\,\mathrm{Fo}\right] d\,\mathrm{Fo}. \tag{17}$$

The form of the dependence $Z_n(\mathrm{Fo})$ may be established by putting

$$\bar{W}(\mathrm{Fo}) = 0, \tag{18}$$

or

$$\left(1-\theta_0+\frac{\mathrm{Po}}{\mu_n^2}\right) \times \left[Z_n' \frac{2J_1(\mu_n)}{\mu_n} - Z_n \mu_n' \left(\frac{4}{\mu_n^2} J_1(\mu_n) - \frac{2}{\mu_n} J_0(\mu_n)\right)\right] - \frac{2\mathrm{Po}\,\mu_n'}{\mu_n^4} Z_n J_1(\mu_n) = 0. \tag{18'}$$

Hence, taking account of (8), we have

$$Z_n(\mathrm{Fo}) = \frac{J_1[\mu_n(0)]}{\mu_n^2(0)} \sqrt{\frac{(1-\theta_0)\mu_n^2(0)+\mathrm{Po}}{(1-\theta_0)\mu_n^2(\mathrm{Fo})+\mathrm{Po}}} \frac{\mu_n^2(\mathrm{Fo})}{J_1[\mu_n(\mathrm{Fo})]}. \tag{19}$$

Carrying out substitution of (17) and (19) into (5), we have, finally,

$$\theta(\psi, \mathrm{Fo}) = 1 + \frac{\mathrm{Po}}{4}\left[1-\psi^2+\frac{2}{\mathrm{Bi}(\mathrm{Fo})}\right] \frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})} \times \exp -\int_0^{\mathrm{Fo}} \frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})} d\,\mathrm{Fo} \times \left\{\frac{4+\mathrm{Bi}(0)}{8\mathrm{Bi}(0)} + \int_0^{\mathrm{Fo}} \left[\exp \int_0^{\mathrm{Fo}} \frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})} d\,\mathrm{Fo}\right] d\,\mathrm{Fo}\right\} - \sum_{n=1}^{\infty} B_n \frac{\sqrt{(1-\theta_0)\mu_n^2(0)+\mathrm{Po}}}{2\mu_n(0)} \frac{\sqrt{(1-\theta_0)\mu_n^2(\mathrm{Fo})+\mathrm{Po}}}{J_1[\mu_n(\mathrm{Fo})]} \times J_0[\mu_n(\mathrm{Fo})\psi] \exp -\int_0^{\mathrm{Fo}} \mu_n^2(\mathrm{Fo})\, d\,\mathrm{Fo}, \tag{5'}$$

where

$$B_n = A_n \frac{2J_1[\mu_n(0)]}{\mu_n(0)}.$$

The values of the roots $\mu_n(\mathrm{Fo})$ for a fixed value of the Bi number may be found with the aid of tables given in [1], which also gives a table of the coefficients B_n. The series in (5') converges rapidly, and therefore, if we exclude small values of Fo, we can confine ourselves to a single term, the first of the series. The form of the functional relation of $\mu_1^2(\mathrm{Fo})$ and Bi(Fo) may be established by expanding the left side of (6) in powers of $\mu_1(\mathrm{Fo})$ and going over to the transformed series [2].

Values of Relative Temperature on the Surface and at the Center of an Infinite Cylinder

Fo	θ (1, Fo) according to (5')	θ (1, Fo) according to [3]	θ (0, Fo) according to (5')	θ (0, Fo) according to [3]
0.25	0.804	0.805	0.650	0.655
0.50	0.949	0.952	0.905	0.903
0.75	1.045	1.048	1.061	1.062
1.00	1.110	1.112	1.162	1.164
1.50	1.188	1.190	1.277	1.279
2.00	1.230	1.233	1.335	1.337

We then have

$$\mu_1^2(\mathrm{Fo}) = 2\mathrm{Bi}(\mathrm{Fo}) - \frac{\mathrm{Bi}^2(\mathrm{Fo})}{2} + \frac{\mathrm{Bi}^3(\mathrm{Fo})}{12} - \ldots. \tag{20}$$

Carrying out a similar transformation of (6'), we obtain

$$\mu_1^2(\mathrm{Fo}) = \frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})} - \frac{1}{192}\left[\frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})}\right]^3 + \ldots. \tag{21}$$

When $\mathrm{Bi}(\mathrm{Fo}) \leq 2.0$, we can confine ourselves to the first term of (20), i. e.,

$$\mu_1^2(\mathrm{Fo}) \approx \frac{8\mathrm{Bi}(\mathrm{Fo})}{4+\mathrm{Bi}(\mathrm{Fo})}. \tag{21'}$$

It should be noted that, from the physical viewpoint, functions (10) and (11) may be regarded as internal heat sources, and relations (14) and (15), respectively, as their mean integral values over the cylinder section. Analysis of the conformity of (5') with differential equation (1) shows that with such averaging the accuracy of the solution proves to be sufficient, while the degree of approximation is the greater, the less the influence of the perturbing term on the value of the parameter Bi(Fo).

As an example we give the calculation of the temperature field in a long cylinder (see table) with the following initial data:

$$\mathrm{Bi}(\mathrm{Fo}) = \frac{16+4\mathrm{Fo}}{12+7\mathrm{Fo}}; \quad \mathrm{Po} = 0.5; \quad \theta_0 = 0.4.$$

For comparison, we also give results of a numerical integration of system (1)–(4) [3].

In conclusion it should be noted that the above method may also be extended to the case of variable medium temperature.

If the parameter Bi remains constant (Bi = const) during heat transfer, then (5') transforms into the exact solution obtained in [1].

NOTATION

$T(r, \tau)$ is the temperature; T_m is the temperature of medium; T_0 is the initial temperature of body; r is a coordinate; τ is the time; a is the thermal diffusivity; λ is the thermal conductivity; $\alpha(\tau)$ is the heat transfer coefficient; R is the cylinder radius; q_ϑ is the specific power of heat source; $\theta(\psi, \mathrm{Fo}) = T(r, \tau)/T_c$; $\theta_0 = T_0/T_c$; $\psi = r/R$; $\mathrm{Fo} = a\tau/R^2$; $\mathrm{Bi}(\mathrm{Fo}) = \alpha(\mathrm{Fo})R/\lambda$; $\mathrm{Po} = q_\vartheta R^2/\lambda T_c$.

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